

A RELATIVE VERSION OF KUMMER THEORY

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ABSTRACT. Let E/F be a cyclic Galois extension of degree p^l with Galois group G . It is shown that the Galois module structure of both sides of the Kummer pairing (for Kummer extensions of E) are the same. In other words, we show that the Kummer duality holds in the level of finitely generated G -modules.

1. INTRODUCTION

Let E/F be a cyclic Galois extension of degree p^l for a prime number p and a positive integer l . We denote the Galois group $\text{Gal}(E/F)$ by G . We also assume F contains a primitive p th root of unity. Let E^\times denote the multiplicative group of E . For a positive integer n , we set $E^{\times n} = \{a^n; a \in E^\times\}$. The Galois action of G on the abelian group $E^\times/E^{\times p}$ gives it an $\mathbb{F}_p[G]$ -module structure. The aim of this paper is to establish an $\mathbb{F}_p[G]$ -module isomorphism between every finitely generated $\mathbb{F}_p[G]$ -submodule of $E^\times/E^{\times p}$ and the Galois group of its associated Kummer extension over E as an $\mathbb{F}_p[G]$ -module. Motivation of this isomorphism comes from certain type of Galois embedding problems. Besides explaining all details, the other advantage of this exposition is that the Galois action of G on the above Galois group is studied explicitly. For applications of this result, we refer the reader to [2] and [4].

2. A RELATIVE VERSION OF KUMMER THEORY

We assume the Galois group $G = \text{Gal}(E/F)$ is generated by σ . We also use notations and definitions of module theory over $\mathbb{F}_p[G]$, developed in [3]. The subgroup of all n th roots of unity in E^\times is denoted by μ_n . If B be a subset of E^\times , then $E_B := E(B^{1/n})$ is the field extension of E obtained by adding all n th roots of elements of B to E . Now, Kummer theory can be summarized in the following theorem:

Theorem 2.1. *Let n be a positive integer and let E be a field whose characteristic does not divide n which contains a primitive n th root of unity. Let B be a subgroup of E^\times containing $E^{\times n}$. Then E_B is Galois over E and the Galois group $N_B = \text{Gal}(E_B/E)$ is abelian of exponent n . Moreover, there is a $\mathbb{Z}/n\mathbb{Z}$ -bilinear map*

$$(2.1) \quad \begin{aligned} N_B \times B/E^{\times n} &\rightarrow \mu_n \\ (\tau, [x]) &\mapsto \langle \tau, [x] \rangle := \tau(\sqrt[n]{x})/\sqrt[n]{x}, \quad \tau \in N_B, x \in B. \end{aligned}$$

The extension E_B/E is finite if and only if $B/E^{\times n}$ is finite, in this case $B/E^{\times n}$ is isomorphic to N_B . Furthermore, the map $B \mapsto E_B$ is a bijection between the set of subgroups of E^\times containing $E^{\times n}$ and Galois extensions of E whose Galois groups over E are abelian of exponent n .

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A field extension of E of the form E_B described in the above theorem is called a *Kummer extension of E of exponent n* . The pairing 2.1 is called *the Kummer pairing*. From now on, we let n be a prime number p . Then, the Kummer pairing is a non-degenerate \mathbb{F}_p -bilinear map. The normal closure of a Kummer extension E_B of E over F is denoted by \overline{E}_B . It is the Kummer extension associated with \overline{B} , the $\mathbb{F}_p[G]$ -submodule of $E^\times/E^{\times p}$ generated by elements of B , see [5]. To describe the action of $G = \text{Gal}(E/F)$ on $\text{Gal}(\overline{E}_B/E)$, or equivalently the $\mathbb{F}_p[G]$ -module structure of $\text{Gal}(\overline{E}_B/E)$, the following remark is useful.

Remark 2.2. There are three Galois actions of G on abelian groups occurring in the Kummer pairing 2.1. Since F contains a primitive p th root of unity, the action of G on μ_p is trivial. It is clear that $E^{\times p}$ is invariant under the Galois action of G . Therefore, if B is a subgroup of E^\times containing $E^{\times p}$ and invariant under the action of G , then $B/E^{\times p}$ is endowed with an action of G defined by $\sigma([a]) := [\sigma(a)]$ for $a \in B$. It also follows from Theorem 1 of [5] that $E_B = \overline{E}_B$, and consequently E_B is Galois over F . In this case, let N_B (resp. H_B) be the Galois group of E_B over E (resp. F), as it is illustrated in the following diagram:

$$(2.2) \quad \begin{array}{c} E_B \\ \downarrow N_B = \text{Gal}(E_B/E) \\ E \\ \downarrow G = \text{Gal}(E/F) \\ F \end{array} \quad \begin{array}{c} \text{---} H_B = \text{Gal}(E_B/F) \text{---} \end{array}$$

Regarding the extension $1 \rightarrow N_B \rightarrow H_B \rightarrow G \rightarrow 1$, let $\tilde{\sigma}$ be a lift of σ in H_B . The action of G on H_B is defined by conjugation, i.e. $\sigma(h) := \tilde{\sigma}h\tilde{\sigma}^{-1}$ for all $h \in H_B$. Since N_B is normal in H_B , this action induces an action of G on N_B . Let σ_0 be another lift of σ in H_B . Then $\sigma_0 = \tilde{\sigma}n_0$ for some $n_0 \in N_B$ and due to the fact that N_B is abelian, for $n \in N_B$, we have $\sigma_0 n \sigma_0^{-1} = \tilde{\sigma} n_0 n n_0^{-1} \tilde{\sigma}^{-1} = \tilde{\sigma} n \tilde{\sigma}^{-1}$. This shows that the above action of G on N_B does not depend on the choice of the lift of σ . Clearly, it does not depend on the choice of the generator of G neither.

Proposition 2.3. ([5]) *The Kummer pairing preserves the Galois module structure of the involving abelian groups. More precisely, for all $b \in B$ and $\tau \in N_B$, we have*

$$(2.3) \quad \langle \sigma(\tau), \sigma([b]) \rangle = \sigma(\langle \tau, [b] \rangle) = \langle \tau, [b] \rangle.$$

Proof. We keep using notations and assumptions of Remark 2.2. Let $b \in B$ and let $y = \sqrt[p]{b}$ be a p th root of b in E_B . Then, we have $\langle \sigma(\tau), \sigma([b]) \rangle = \langle \sigma(\tau), [\sigma(b)] \rangle = \frac{\tilde{\sigma}\tau\tilde{\sigma}^{-1} \sqrt[p]{\sigma(b)}}{\sqrt[p]{\sigma(b)}} = \frac{\tilde{\sigma}\tau\tilde{\sigma}^{-1} \sqrt[p]{\sigma(y^p)}}{\sqrt[p]{\sigma(y^p)}} = \frac{\tilde{\sigma}\tau\tilde{\sigma}^{-1} \sqrt[p]{(\tilde{\sigma}(y))^p}}{\sqrt[p]{(\tilde{\sigma}(y))^p}} = \frac{\tilde{\sigma}\tau\tilde{\sigma}^{-1}\tilde{\sigma}(y)}{\tilde{\sigma}(y)} = \frac{\tilde{\sigma}\tau(y)}{\tilde{\sigma}(y)} = \tilde{\sigma} \left(\frac{\tau(\sqrt[p]{b})}{(\sqrt[p]{b})} \right) = \tilde{\sigma}(\langle \tau, [b] \rangle) = \sigma(\langle \tau, [b] \rangle) = \langle \tau, [b] \rangle$. The last equality follows from the fact that the action of G on μ_p is trivial. \square

Our next goal is to show that if $B/E^{\times p}$ is a finitely generated $\mathbb{F}_p[G]$ -submodule of $E^\times/E^{\times p}$, then the $\mathbb{F}_p[G]$ -module structures of $B/E^{\times p}$ and N_B are the same. Here, our approach is based on the structure of N_B as a Galois group with a G -action. Let $a \in E^\times$. In the following remarks, we make some observations about the group structure and the $\mathbb{F}_p[G]$ -module structure of $\text{Gal}(\overline{E(a^{1/p})}/E)$.

Remarks 2.4. Let s be the smallest natural number such that $x^s a$ is a p th power in E .

- (i) It is clear from Theorem 1 of [5] that $\overline{E(a^{1/p})}$ is a splitting field of the following family of polynomials in $E[y]$

$$\{P_0(y) = y^p - a, P_1(y) = y^p - xa, \dots, P_{s-1}(y) = y^p - x^{s-1}a\},$$

where $x = \sigma - 1$ as in [3]. Let ζ be a primitive p th root of unity, and let α_0 be a root of $P_0(y)$. Then, for $i = 1, \dots, s-1$, we define $\alpha_i := x^i \alpha_0$. Now, it is easy to see that for $i = 0, \dots, s-1$, the roots of $P_i(y)$ are $\alpha_i, \zeta \alpha_i, \dots, \zeta^{p-1} \alpha_i$. Furthermore, we note that an arbitrary element (n_0, \dots, n_{s-1}) in $(\mathbb{Z}/p\mathbb{Z})^s \simeq \text{Gal}(\overline{E(a^{1/p})}/E)$ permutes these roots by the formula $(n_0, \dots, n_{s-1})(\zeta^j \alpha_i) = \zeta^{j+n_i} \alpha_i$.

- (ii) Since $x^s a$ is a p th power in E , we have $(x \alpha_{s-1})^p = x(\alpha_{s-1})^p = x(x^{s-1}a) = x^s a = b^p$ for some $b \in E$. Thus, $x \alpha_{s-1}$ is a p th root of b^p . On the other hand b is a p th root of b^p too. So, $x \alpha_{s-1} = \zeta^k b$ for an integer k and consequently $x \alpha_{s-1} \in E$. Therefore, $x^j \alpha_i \in E$, if $i+j \geq s$. On the other hand, by our definition, we have $x^j \alpha_i = \alpha_{i+j}$, whenever $i+j < s$.

We know from [3] that $\mathbb{F}_p[G]$, as a ring, is isomorphic to $\mathbb{F}_p[x]/x^q$, where $q = p^l$.

Lemma 2.5. *With the above notation, let $\rho : A \rightarrow \text{Gal}(\overline{E(a^{1/p})}/E)$ as an $\mathbb{F}_p[G]$ -homomorphism be defined by $\rho(1) = (0, \dots, 0, 1)$. Then, the kernel of ρ is $\langle x^s \rangle$.*

Proof. By Proposition 1.2 of [3], the kernel of ρ is of the form $\langle x^m \rangle$ for some $m \leq q$. If $m > s$, then we obtain an injective map from $A/\langle x^m \rangle$ into $\text{Gal}(\overline{E(a^{1/p})}/E) \simeq (\mathbb{Z}/p\mathbb{Z})^s$, which is impossible because it implies that p^m , the cardinality of $A/\langle x^m \rangle$, is less than or equal to p^s , the cardinality of $(\mathbb{Z}/p\mathbb{Z})^s$. Now, by showing that $\rho(x^{s-1}) \neq \text{id}$, we prove that m cannot be less than s . For $0 \leq n \leq s-2$, let $\rho(x^n) = (n_0, \dots, n_{s-1})$ and let $\rho(x^{n+1}) = (m_0, \dots, m_{s-1})$.

Claim: Let n_i be the last non-zero component of $\rho(x^n)$. If $i > 0$, then $m_{i-1} \neq 0$. To see this, we have to compute $\rho(x^{n+1})(\alpha_{i-1})$. With the notations of Remark 2.2, we have $\rho(x^{n+1})\rho(x^n) = \rho(x^{n+1} + x^n) = \rho((x+1)(x^n)) = (x+1)\rho(x^n) = \tilde{\sigma}(\rho(x^n)) = \tilde{\sigma}\rho(x^n)\tilde{\sigma}^{-1} = (x+1)\rho(x^n)(x+1)^{-1}$. Thus, we have

$$(2.4) \quad \rho(x^{n+1}) = (x+1)\rho(x^n)(x+1)^{-1}\rho(x^n)^{-1}.$$

Hence, $\rho(x^{n+1})(\alpha_{i-1}) = (x+1)\rho(x^n)(x+1)^{-1}\rho(x^n)^{-1}(\alpha_{i-1}) = (x+1)\rho(x^n)(x+1)^{-1}(\zeta^{-n_{i-1}}\alpha_{i-1}) = \zeta^{-n_{i-1}}(x+1)\rho(x^n)(1-x+x^2-\dots)(\alpha_{i-1}) = \zeta^{-n_{i-1}}(x+1)\rho(x^n)\left(\frac{\alpha_{i-1}(x^2\alpha_{i-1})\dots}{(x\alpha_{i-1})(x^3\alpha_{i-1})\dots}\right) = \zeta^{-n_{i-1}}(x+1)\rho(x^n)\left(\frac{\alpha_{i-1}(x^2\alpha_{i-1})\dots}{\alpha_i(x^3\alpha_{i-1})\dots}\right) = \star$. We note that for $r > 1$, $x^r \alpha_{i-1}$ is either α_{i+r-1} or an element of E , see Remarks 2.4. In both cases, $\rho(x^n)$ does not change it. Thus, we have $\star = \zeta^{-n_{i-1}}(x+1)\left(\frac{\zeta^{n_{i-1}-1}\alpha_{i-1}(x^2\alpha_{i-1})\dots}{\zeta^{n_i}\alpha_i(x^3\alpha_{i-1})\dots}\right) = \zeta^{-n_i}(x+1)\left(\frac{\alpha_{i-1}(x^2\alpha_{i-1})\dots}{\alpha_i(x^3\alpha_{i-1})\dots}\right) = \zeta^{-n_i}(x+1)((x+1)^{-1}(\alpha_{i-1})) = \zeta^{-n_i}\alpha_{i-1}$. Therefore, $m_{i-1} = -n_i \neq 0$.

Now, since the component with index $s-1$ in $\rho(1)$ is non-zero, the component with index $s-2$ in $\rho(x)$ is non-zero and one can argue by induction that $\rho(x^k)$ has a non-zero component for $0 \leq k < s$. Thus, $\rho(x^k) \neq \text{id}$ for $0 \leq k < s$. \square

We call the following theorem the *relative Kummer theory*.

Theorem 2.6. *Let E/F be a cyclic extension of degree $q = p^l$ with Galois group G and let F contain a primitive p th root of unity. Let B be a subgroup of E^\times containing $E^{\times p}$ and invariant under the Galois action of G . If $B/E^{\times p}$ is a finitely generated $\mathbb{F}_p[G]$ -module, it has the same $\mathbb{F}_p[G]$ -module structure as $N_B = \text{Gal}(E_B/E)$.*

Proof. We prove the statement by induction on the number of summands in the decomposition of $B/E^{\times p}$ into direct sum of cyclic modules. In the case $B/E^{\times p} = 0$ there is nothing to prove. So, let $B/E^{\times p} \simeq B_1/E^{\times p} \oplus \cdots \oplus B_n/E^{\times p}$ be a decomposition of $B/E^{\times p}$ into direct sum of cyclic modules, and let the statement of the theorem be true for $\mathbb{F}_p[G]$ -submodules of $E^\times/E^{\times p}$ whose decompositions have less summands than n .

For $i = 1, \dots, n$, let $\dim_{\mathbb{F}_p} B_i/E^{\times p} = l_i$. We choose $b_i \in B_i$ such that $[b_i]$ is a generator of $B_i/E^{\times p}$ as an $\mathbb{F}_p[G]$ -module. Consider

$$B[1] = B_2/E^{\times p} \oplus \cdots \oplus B_n/E^{\times p}$$

as a submodule of $B/E^{\times p}$, and let $N[1] = \text{Gal}(E_B/E_{B[1]})$, as illustrated in the following diagram:

$$(2.5) \quad \begin{array}{c} E_B \\ \downarrow N[1] \\ E_{B[1]} \\ \downarrow N_{B[1]} \\ E \end{array} \quad \begin{array}{c} \left. \begin{array}{c} \downarrow N_B \\ \downarrow N_{B[1]} \end{array} \right\} \end{array}$$

To prove that N_B and $B/E^{\times p}$ have the same $\mathbb{F}_p[G]$ -module structure, we first show that $N[1] \simeq B_1/E^{\times p}$. Let

$$C = \{x^{j_i} b_i; i = 1, \dots, n \text{ and } 0 \leq j_i \leq l_i - 1\}.$$

By Theorem 1 of [5], $E_B = E(C^{1/p})$. Therefore,

$$E_B = E_{B[1]}(b_1^{1/p}, (xb_1)^{1/p}, \dots, (x^{l_1-1}b_1)^{1/p}).$$

Let $a = b_1$ and $s = l_1$. Then, Remarks 2.4 applies to the extension $E_B = E_{B[1]}(a^{1/p})$ over $E_{B[1]}$ and similar to Lemma 2.5, we define an $\mathbb{F}_p[G]$ -homomorphism $\rho : A \rightarrow N[1]$ whose kernel is $\langle x^s \rangle$. This map gives rise to an injective $\mathbb{F}_p[G]$ -homomorphism, say $\tilde{\rho}$ of $A/\langle x^s \rangle \simeq B_1/E^{\times p}$ into $N[1] = \text{Gal}(E_B/E_{B[1]}) = \mathbb{Z}/p\mathbb{Z}^s$. Since the source and the target of $\tilde{\rho}$ have the same cardinality, it is an $\mathbb{F}_p[G]$ -isomorphism. Therefore,

$$(2.6) \quad N[1] \simeq B_1/E^{\times p}.$$

Now, we claim that $N[1]$ is an $\mathbb{F}_p[G]$ -submodule of N_B . To see this, let $\tau \in N[1]$. Since $B[1]$ is an $\mathbb{F}_p[G]$ -submodule of $B/E^{\times p}$, for every $b \in B[1]$, $\sigma^{-1}b \in B[1]$. Thus, by Proposition 2.3, for every $b \in B[1]$ we have $\langle \sigma\tau, b \rangle = \langle \sigma\tau, \sigma\sigma^{-1}b \rangle = \langle \tau, \sigma^{-1}b \rangle = 1$. This implies that $\sigma\tau \in N[1]$.

On the other hand, by Kummer theory, we have the following decomposition of N_B into a direct sum of two abelian subgroups:

$$(2.7) \quad N_B \simeq N[1] \oplus N_{B[1]}.$$

We claim that $N_{B[1]}$ is an $\mathbb{F}_p[G]$ -submodule of N_B . To see this, consider an element of the basis of E_B over $E_{B[1]}$, say $(x^j b_1)^{1/p}$. Then, for $\tau \in N_{B[1]}$, we have

$\sigma(\tau)(x^j b_1)^{1/p} = \sigma\tau\sigma^{-1}(x^j b_1)^{1/p} = (x+1)\tau(x+1)^{-1}(x^j b_1)^{1/p} = (x+1)\tau(1-x+x^2-\dots)(x^j b_1)^{1/p} = \star$. Now, one notes that every term of $(1-x+x^2-\dots)(x^j b_1)^{1/p}$ is again an element of the basis of E_B over $E_{B[1]}$, so τ acts trivially on it and we have $\star = (x+1)(1-x+x^2-\dots)(x^j b_1)^{1/p} = (x+1)(x+1)^{-1}(x^j b_1)^{1/p} = (x^j b_1)^{1/p}$. This shows that $\sigma\tau \in N_{B[1]}$. Therefore, 2.7 can be considered as a decomposition of N_B into a direct sum of two $\mathbb{F}_p[G]$ -submodules. By the induction hypothesis, $N_{B[1]} \simeq B[1]$. Therefore, using 2.6 and 2.7 we have $N_B \simeq B_1/E^{\times p} \oplus B[1] = B/E^{\times p}$ as two $\mathbb{F}_p[G]$ -modules. \square

3. THE SECOND PROOF

In this section, we explain another approach to prove the relative Kummer theory. This approach has been suggested to the author by Mináč, and was appeared concisely in [2]. Here, we first use the fact that the Kummer pairing preserves the G -action, Proposition 2.3. Then, we prove an analogue of Lemma 2.5 to show cyclic $\mathbb{F}_p[G]$ -modules are self-dual. The advantage of this approach is its simplicity, also it is shorter. However, one does not see the $\mathbb{F}_p[G]$ -module structure of N_B concretely here.

It is well-known that the Kummer pairing defines a group isomorphism between N_B and $\text{Hom}_{\mathbb{Z}}(B/E^{\times p}, \mathbb{F}_p)$ as follows:

$$(3.1) \quad \begin{aligned} \Psi : N_B &\rightarrow \text{Hom}_{\mathbb{Z}}(B/E^{\times p}, \mathbb{F}_p) \\ \tau &\mapsto \Psi_\tau : \begin{array}{ccc} B/E^{\times p} & \rightarrow & \mathbb{F}_p \\ [b] & \mapsto & \langle \tau, [b] \rangle \end{array} \end{aligned}$$

The following definition describes the $\mathbb{F}_p[G]$ -module structure of the abelian group $\text{Hom}_{\mathbb{Z}}(B/E^{\times p}, \mathbb{F}_p)$, the interested reader can find more details about this in [1], page 48.

Definition 3.1. Let a group G act on two abelian groups H and K . Then, G acts on $\text{Hom}_{\mathbb{Z}}(H, K)$ as follows:

$$g\theta(h) = g\theta(g^{-1}h),$$

where $g \in G$, $\theta \in \text{Hom}_{\mathbb{Z}}(H, K)$ and $h \in H$.

One notes that the above $\mathbb{F}_p[G]$ -module structure on $\text{Hom}_{\mathbb{Z}}(H, K)$ is different from its usual module structure as the Hom functor of two $\mathbb{F}_p[G]$ -modules. We consider only this $\mathbb{F}_p[G]$ -module structure in this paper.

Remark 3.2. First, we note that since $B/E^{\times p}$ is of exponent p , every homomorphism in $\text{Hom}_{\mathbb{Z}}(B/E^{\times p}, \mathbb{F}_p)$ can be considered mod p . So, this Hom is the same as $\text{Hom}_{\mathbb{F}_p}(B/E^{\times p}, \mathbb{F}_p)$, which is the dual of $B/E^{\times p}$ as a vector space over \mathbb{F}_p . Now, by considering the above $\mathbb{F}_p[G]$ -module structure on $\text{Hom}_{\mathbb{F}_p}(B/E^{\times p}, \mathbb{F}_p)$, it is easy to see that Ψ as defined in 3.1 is an $\mathbb{F}_p[G]$ -isomorphism. To see this, let $\sigma \in G$ and $\tau \in N_B$, then we have $[\Psi(\sigma\tau)]([b]) = \langle \sigma\tau, [b] \rangle = \langle \tau, \sigma^{-1}[b] \rangle = \langle \sigma\tau, \sigma\sigma^{-1}[b] \rangle = \langle \tau, \sigma^{-1}[b] \rangle = [\Psi(\tau)](\sigma^{-1}[b]) = \sigma([\Psi(\tau)](\sigma^{-1}[b]))$. Since F contains a p th root of unity, we have $[\Psi(\sigma\tau)]([b]) = [\sigma(\Psi(\tau))]([b])$ for every $b \in B$.

Proposition 3.3. Let M be a finitely generated $\mathbb{F}_p[G]$ -module. Then M and $\text{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p)$ are isomorphic as $\mathbb{F}_p[G]$ -modules.

Proof. If $M \simeq M_1 \oplus M_2$, it is easy to see that, for $i = 1, 2$, $\text{Hom}_{\mathbb{F}_p}(M_i, \mathbb{F}_p)$ can be considered as an $\mathbb{F}_p[G]$ -submodule of $\text{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p)$. Thus $\text{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p) \simeq \text{Hom}_{\mathbb{F}_p}(M_1, \mathbb{F}_p) \oplus \text{Hom}_{\mathbb{F}_p}(M_2, \mathbb{F}_p)$ as $\mathbb{F}_p[G]$ -modules. Therefore it is enough to prove the statement in the case that M is cyclic. Let $\dim_{\mathbb{F}_p} M = l$. Regarding the automorphism of $\mathbb{F}_p[G]$ induced by the map $\sigma \mapsto \sigma^{-1}$, it is clear that the ideal generated by $(\sigma^{-1} - 1)^l$ is isomorphic with the ideal generated with $(\sigma - 1)^l$. Thus, one can consider M as $\mathbb{F}_p[G]/\langle y^l \rangle$, where $y = \sigma^{-1} - 1$. We define $f \in \text{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p)$ by extending rules $f(1) = f(x) = \cdots = f(x^{l-2}) = 0$ and $f(x^{l-1}) = 1$ linearly to M . It follows from the $\mathbb{F}_p[G]$ -module structure of $\text{Hom}_{\mathbb{F}_p}(M, \mathbb{F}_p)$ that $((\sigma - 1)f)(m) = f((\sigma^{-1} - 1)m)$ for $m \in M$. By repeating this we get $((\sigma - 1)^{l-1}f) = f(y^{l-1}) = 1$. Hence, we obtain $\dim_{\mathbb{F}_p}(\langle f \rangle) \geq l$. Since $\text{Hom}_{\mathbb{F}_p}(M_1, \mathbb{F}_p)$ has dimension l over \mathbb{F}_p , we conclude that $\text{Hom}_{\mathbb{F}_p}(M_1, \mathbb{F}_p) = \langle f \rangle$. Thus, both M and $\text{Hom}_{\mathbb{F}_p}(M_1, \mathbb{F}_p)$ are cyclic $\mathbb{F}_p[G]$ -modules of dimension l , and so they are isomorphic. \square

It is clear that Theorem 2.6 follows from Remark 3.2 and the above proposition.

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